4.5 – Coordinates and Basis

Definition: If $S = \{\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_n\}$ is a set of vectors in a finite-dimensional vector space V, then S is called a **basis** for V if:
a) S spans V.
b) S is linearly independent.

$\{\vec{e}_1, \vec{e}_2, \vec{e}_3\}$ are standard basis vectors for R^3. These relate to mutually perpendicular axes. However, any 3 vectors in R^3 that do not lie in the same plane can be used to represent any vector in R^3. These 3 vectors are called basis vectors.

#1 Use the determinant of a coefficient matrix to show that the following set of vectors forms a basis for R^2: $\{(2,\ 1),\ (3,\ 0)\}$. $\{(2,1),(3,0)\} = \{(3,0),(2,1)\}$

Let $A = \begin{bmatrix} 2 & 3 \\ 1 & 0 \end{bmatrix}$. If $A\vec{x} = \vec{b}$ always has a solution, then the set spans R^2. If $A\vec{x} = \vec{0}$ has only the trivial solution, then the set is independent.

$\det(A) = -3 \neq 0$ so both conditions are met $\Rightarrow$ the set is a basis.

#8 Show that the following vectors do not form a basis for P_2.

$1 - 3x + 2x^2$, $1 + x + 4x^2$, $1 - 7x$

$2 - 28 + 0 = -26$

$$\begin{vmatrix} 1 & 1 & 1 \\ -3 & 1 & -7 \\ 2 & 4 & 0 \end{vmatrix} \begin{matrix} 1 & 1 \\ -3 & 1 \\ 2 & 4 \end{matrix}$$

$-26 - (-26) = 0$

$0 - 14 - 12 = -26$

The set is dependent and so is not a basis.

Note that $2(1-3x+2x^2) - (1+x+4x^2) = 1-7x$

Definition: A basis in which the listed order of the vectors matters is called an **ordered basis**.

Some Standard bases

$\{\hat{i},\ \hat{j}\}$ is a basis for R^2 (this is the same as $\{\mathbf{e}_1, \mathbf{e}_2\}$).

$\{\hat{i},\ \hat{j},\ \hat{k}\}$ is basis for R^3 (this is the same as $\{\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3\}$).

$\{\mathbf{e}_1, \mathbf{e}_2, \ldots, \mathbf{e}_n\}$ is a basis for R^n.

$\{1,\ x,\ x^2,\ \ldots,\ x^n\}$ is a basis for P_n.

$\left\{ \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix} \right\}$ is a basis for M_{22}.

The basis for the zero vector space $\vec{V} = \{\vec{0}\}$ is $\emptyset$ (the empty set).

Theorem 4.5.1 Uniqueness of Basis Representation

If $S = \{\mathbf{v}_1,\ \mathbf{v}_2,\ \ldots,\ \mathbf{v}_n\}$ is a basis for a vector space V, then every vector $\mathbf{v}$ in V can be expressed in the form $\mathbf{v} = c_1\,\mathbf{v}_1 + c_2\,\mathbf{v}_2 + \ldots + c_n\,\mathbf{v}_n$ in exactly one way.

at least 1

not more than 1

Pf: Let $\vec{v} \in V$. Since S spans V, $\exists\, c_1, c_2, \dots, c_n$ such that $\vec{v} = c_1\vec{v}_1 + c_2\vec{v}_2 + \dots + c_n\vec{v}_n$.

Suppose there are $k_1, k_2, \dots, k_n$ such that

$$\vec{v} = k_1\vec{v}_1 + k_2\vec{v}_2 + \dots + k_n\vec{v}_n$$

$$\vec{v} - \vec{v} = 0 = c_1\vec{v}_1 + c_2\vec{v}_2 + \dots + c_n v_n - (k_1\vec{v}_1 + k_2\vec{v}_2 + \dots + k_n\vec{v}_n)$$

$$\Rightarrow (c_1 - k_1)\vec{v}_1 + (c_2 - k_2)\vec{v}_2 + \dots + (c_n - k_n)\vec{v}_n = \vec{0}$$

Since S is independent, $c_i - k_i = 0$ for $1 \le i \le n$

$$\Rightarrow c_i = k_i, \quad 1 \le i \le n.$$

There is only one way to represent $\vec{v}$ using S.

Definition: If $S = \{\mathbf{v}_1, \mathbf{v}_2, \dots, \mathbf{v}_n\}$ is an ordered basis for a vector space V, and $\mathbf{v} = c_1\mathbf{v}_1 + c_2\mathbf{v}_2 + \dots + c_n\mathbf{v}_n$ is the expression for a vector $\mathbf{v}$ in terms of the basis S, then the scalars $c_1, c_2, \dots, c_n$ are called the **coordinates of v relative to the basis** S. The vector $(c_1, c_2, \dots, c_n)$ in R^nconstructed from these coordinates is called the **coordinate vector of v relative to the basis** S; it is denoted by $(\mathbf{v})_S = (c_1, c_2, \dots, c_n)$.

$$[\vec{v}]_S = \begin{bmatrix} c_1 \\ c_2 \\ \vdots \\ c_n \end{bmatrix}$$

Example: Consider $\mathbf{v} = (-1, 7, 2) \in R^3$. The set $B = \{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3\}$, where $\mathbf{v}_1 = (2, -1, -1)$, $\mathbf{v}_2 = (-2, 1, 2)$, and $\mathbf{v}_3 = (3, 5, 4)$ forms a basis for R^3 (verify). Find $(\mathbf{v})_S$ (where S is the standard basis for R^3) and $(\mathbf{v})_B$.

$-3 + 20 + 8 = 25$

$$\begin{vmatrix} 2 & -2 & 3 \\ -1 & 1 & 5 \\ -1 & 2 & 4 \end{vmatrix} \begin{matrix} 2 & -2 \\ -1 & 1 \\ -1 & 2 \end{matrix}$$

$8 + 10 - 6 = 12$

$12 - 25 = -13 \neq 0$ ✓

Want $c_1, c_2, c_3 \ni c_1\vec{v}_1 + c_2\vec{v}_2 + c_3\vec{v}_3 = \vec{v}$

$$\left[\begin{array}{ccc|c} 2 & -2 & 3 & -1 \\ -1 & 1 & 5 & 7 \\ -1 & 2 & 4 & 2 \end{array}\right] \to \left[\begin{array}{ccc|c} 2 & -2 & 3 & -1 \\ 0 & 2 & 11 & 3 \\ 0 & 0 & 1 & 1 \end{array}\right] \to \left[\begin{array}{ccc|c} 1 & -1 & 0 & -2 \\ 0 & 1 & 0 & -4 \\ 0 & 0 & 1 & 1 \end{array}\right]$$

$R_2 \to R_1 + 2R_2$

$$\begin{array}{cccc} 2 & -2 & 3 & -1 \\ -2 & 2 & 10 & 14 \\ \hline 0 & 0 & 13 & 13 \end{array}$$

$R_3 \to R_1 + 2R_3$

$$\begin{array}{cccc} 2 & -2 & 3 & -1 \\ -2 & 4 & 8 & 4 \\ \hline 0 & 2 & 11 & 3 \end{array}$$

$R_2 \to R_2 - 11R_3$

$$\begin{array}{cccc} 0 & 2 & 11 & 3 \\ 0 & 0 & -11 & -11 \\ \hline 0 & 2 & 0 & -8 \end{array}$$

$R_1 \to R_1 - 3R_3$

$$\begin{array}{cccc} 2 & -2 & 3 & -1 \\ 0 & 0 & -3 & -3 \\ \hline 2 & -2 & 0 & -4 \end{array}$$

$R_1 \to R_1 + R_2$

$$\left[\begin{array}{ccc|c} 1 & 0 & 0 & -6 \\ 0 & 1 & 0 & -4 \\ 0 & 0 & 1 & 1 \end{array}\right]$$

so $c_1 = -6$, $c_2 = -4$, $c_3 = 1$

$$\underline{-6}\,\vec{v}_1 \underline{-4}\,\vec{v}_2 \underline{+}\,\vec{v}_3 = \vec{v}$$

check: $-6(2,-1,-1) - 4(-2,1,2) + (3,5,4)$

$= (-12, 6, 6) + (8, -4, -8) + (3, 5, 4)$

$= (-1, 7, 2) = \vec{v}$ ✓

$$(\vec{v})_B = (-6, -4, 1)$$

But using $S = \{\vec{e}_1, \vec{e}_2, \vec{e}_3\}$ we have

$(\vec{v})_S = (-1, 7, 2) = \vec{v}$ Note: $(\vec{v})_S = \vec{v}$ always when S is a standard basis for R^n.

is in $\mathbb{R}^3$

$(\vec{v})_B = (-6, -4, 1) \longrightarrow \vec{v} = (-1, 7, 2)$ is in the vector space V

$(\vec{v})_S = (-1, 7, 2)$

#16 First show that the set $S = \{A_1,\ A_2,\ A_3,\ A_4\}$ is a basis for M_{22}, then express A as a linear combination of the vectors in S, and then find the coordinate vector of A relative to S.

$$A_1 = \begin{bmatrix} 1 & 0 \\ 1 & 0 \end{bmatrix},\ A_2 = \begin{bmatrix} 1 & 1 \\ 0 & 0 \end{bmatrix},\ A_3 = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix},\ A_4 = \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix};\ A = \begin{bmatrix} 6 & 2 \\ 5 & 3 \end{bmatrix}$$

$$\begin{array}{cccc} \vec{A}_1 & \vec{A}_2 & \vec{A}_3 & \vec{A}_4 \end{array}$$
$$\begin{bmatrix} 1 & 1 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{bmatrix} \to \begin{bmatrix} 1 & 1 & 1 & 0 \\ 0 & 1 & 1 & -1 \\ 0 & 0 & 1 & -1 \\ 0 & 0 & 0 & -1 \end{bmatrix}$$

There's a pivot in every column so rref is I.

$\{A_1, A_2, A_3, A_4\}$ is a basis for M_{22}.

Recall that for M_{22}, $B = \left\{\begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}\right\}$.

Thus, $(A)_B = (6, 2, 5, 3)$.

Standard basis

This is in $\mathbb{R}^4$. Note $A \in M_{22}$ $\leftarrow V$

$$\begin{bmatrix} c_1 & 0 \\ c_1 & 0 \end{bmatrix} + \begin{bmatrix} c_2 & c_2 \\ 0 & 0 \end{bmatrix} + \begin{bmatrix} c_3 & 0 \\ 0 & c_3 \end{bmatrix} + \begin{bmatrix} 0 & 0 \\ c_4 & 0 \end{bmatrix} = \begin{bmatrix} 6 & 2 \\ 5 & 3 \end{bmatrix}$$

$A_1 = \begin{bmatrix} 1 & 0 \\ 1 & 0 \end{bmatrix}$, $A_2 = \begin{bmatrix} 1 & 1 \\ 0 & 0 \end{bmatrix}$, $A_3 = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$, $A_4 = \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix}$; $A = \begin{bmatrix} 6 & 2 \\ 5 & 3 \end{bmatrix}$

$$c_1A_1 + c_2A_2 + c_3A_3 + c_4A_4 = A$$

$c_2 = 2$ $\qquad$ $c_3 = 3$

$c_1 + c_2 + c_3 = 6 \Rightarrow c_1 = 1$ $\qquad$ $c_1 + c_4 = 5 \Rightarrow c_4 = 4$

$$(A)_S = (1, 2, 3, 4)$$